

The one example of Lorentz group

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Abstract

The aim of this work is to show, on the example of the behaviour of the spinless charged particle in the homogeneous electric field ,that one can quantized the velocity of particle by the special gauge fixation. The work gives also the some information about the theory of second quantisation in the space of Hilbert-Fock and the theory of projectors in the Hilbert space. One consider in Appendix the theory of the spinless charged particle in the homogeneous addiabatical changed electrical field.

1.Introduction

The history of physics in xx century is determined with the two great discoveries. This is the A. Einstein's relativity theory (the special and the general relativity) and the quantum mechanics. The hypothesis about the discrete nature of light, which was suggested by M.K.E.L. Planck in 1900 y. (as a 'saving' from the *ultra-violet catastrophe*: the endless character of the energy density of the radiation spectrum because of the contribution of the ultra-violet part of the spectrum) led M. Planck (in fact against his 'classical education' as a physicist of xix century with its Maxwell electrodynamics and Newton-Lagrange mechanics) to the conception of the *quantum of minimal action*, i.e. to the Planck constant \hbar . This was a greatest drama of doctor's Planck life: he could not accept up to his death in 1947 y. the 'quantum revolution'.

What is this- the quantum of minimal action? Our reader, which acquainted with classical Lagrange mechanics, is knows that [1] one can represent the phase space Γ as a

$$\Gamma = \int dq_1 \dots dq_s dp_1 \dots dp_s \quad (1)$$

where q_i, p_i are the canonical co-ordinates and the canonical momentum correspondingly. The M. Planck's hypothesis, in the terms of formula (1), signifies that *the cell of the phase space with the size $\Gamma \leq (2\pi\hbar)^3$ is not exists!* This conclusion (at least for the quasiclassical approach in quantum mechanics) signifies non other then *Heisenberg inequality*-the fundamental law of quantum mechanics! So, on author's of this article opinion, an opportunity for the quantum mechanics appearance was as long ago in 1900 y. (with account of the accumulated knowledge in geometrical optics: the Fermat principle, -the law of minimal action in geometrical optics, and the optical-mechanical analogy [2]). In fact, one can interpret geometrical optics as a *quasiclassic theory of photon!*

Albert Einstein as it well known also could not accept quantum mechanics (by the of many years friendship with N. Bohr. Their discussions, the examples and the counterexamples 'for' and 'against' of the quantum theory, -all this had often the highly stormy nature!)

Indeed, the 'precipice' between these two great theories is not large. There are (quantum mechanics and the relativity theory)- the two corner-stones, on which is based the majestic building of modern theoretical physics. One of examples -the quantum -field theories: quantum electrodynamics, QCD, the electroweak theory. They all are the relativistic quantum theories.

The basic postulates of the quantum -field theory, - *Wightman axioms* [3], are based on Lorentz-Poincare group in Minkowski space. The demand of Poincare invariantness of *Wightman functions* (the vacuum averages of the operator product of some quantum-fields), i.e. the demand of relativistic invariantness of S- matrix and the effective section; *the microcause principle*, i.e. zero value of the fields (anti)commutator at the space-like interval between them, - there are the basic displays of special relativity in the quantum -field theory.

The cause of such large difference between quantum mechanics and 'old' Lagrange-Newton (classical) mechanics is the Planck constant \hbar . The introduction of the quantum \hbar in theoretical physics lead to the replacement of *Poisson brackets* onto the *commutation* relation between the canonical co-ordinates and the canonical momentum. These *canonical commutation relations* turn

into Poisson brackets of Hamilton mechanics in the limit of $\hbar=0$ [4]. **The construction of the eigenfunctions of the quantum operators (i.e. of the vectors of Hilbert space):in particular for the quantum-field operators, which we consider as a canonical co-ordinates or a canonical momentum; the solution of the movement equations (Schrodinger equation in non-relativistic quantum mechanics or the quantum-field equations (Dirac equation,Klein-Gordan-Fock equation, Rarita-Schwinger equation,- are the basic examples of the relativistic equations),- are the basic problems of the quantum theory).** This is very important that the quantum- field operators have the dimension of the canonical co-ordinates or the canonical momentum.

One writes down the movement equations, issuing from the classical Lagrangian of corresponding particle and the Lagrange equations as a conditions of the minimal action (for example- the case of the scalar particle).Then we replace all *classical* momenta on the momentum operators $-i\hbar\partial/\partial x_i$. One calls all this process as the *first quantization* (it is appropriate to mention here that the conclusion of Dirac equation is based on Schrodinger equation and on the superposition principle (the basic principle of quantum mechanics) which dictates the form of energy operator (the quantum-mechanical generalization of Hamiltonian):this operator is proportional to Dirac matrixes[5].Thus we obtain this equation from the *quantum-mechanical principles*. But we can conclude it as Lagrange equation [6]).

The second quantization this is ([7],[3]) the point of view onto the quantum field as a *multi-particle field*.The every particle we then interpret as a *one degree of freedom* . Then we construct *Hilbert-Fock space of second quantization*.

The form of the eigenvectors of Hilbert-Fock space is given, for example, in monograph [3] (look formula (7.99)).This form depends on helicity s (explicitly and through the spinors $\omega_{\alpha i}$. On the language of Penrose *flag* structure of the space-time, the spinors $\omega_{\alpha i}$ are [8] *the main spinors* ,by which we decompose arbitrary spin-tensor

$$\psi(\omega, \varpi) = \sum_{\alpha_1 \dots \alpha_{2j}; \beta_1 \dots \beta_{2k'}} \omega^{\alpha_1} \dots \omega^{\alpha_{2j}} \varpi^{\beta_1} \dots \varpi^{\beta_{2k'}} \quad (2)$$

Thus formula (7.99) in[3] is in fact the usual decomposition of the wave function on the space and the spinor parts (in the momentum representation)

The structure of Hilbert space is determined with[3] *Gelfand-Naimark-Segal*

(GNS)construction. Let some algebra with involution U is given. Let us denote this algebra as C^* .Then exists the isomorphism π of algebra U into algebra $B(H)$ of all linear limited operators in the Hilbert space.The representation π is called *unreducible* if every closed subspace in H , which is invariant relatively to all operators $\pi(A); A \in U$ is \emptyset or all H .The vector $\Phi \in H$ is called *the cyclical vector* for representation π if all vectors of the form $\pi(A)\Phi$ where $A \in U$ form the total set in H (correspondingly , the such representation with the cyclical vector is called *cyclical*).

If Φ is the vector in H then it generates the *positive functional*

$$F_\Phi = \langle \Phi, \pi(A)\Phi \rangle \quad (3)$$

on U (in the terms of the probability theory it is the *mathematical expectation* of the value $\pi(A)$ *in state* Φ).This functional is called *the vector functional* , associated with representation π and vector Φ .

In these terms GNS construction consists in following:

One can determine some (cyclical) representation π_F of algebra U in the Hilbert space with cyclical vector Φ_F for given positive functional F such that

$$F(A) = \langle \Phi_F, \pi_F(A) \Phi_F \rangle \quad (4)$$

The representation π_F is determined with these conditions unique with precision of the unitary equivalence.

The one of the most important for the quantum-field theory are *birth and annihilation operators*. As it is well known, the spin of the particle determines the two kinds of statistics : *the Bose-Einstein statistics for the integer spin and the Fermi statistics for the semi-integer spin* (the Maxwell statistics is the limit of these statistics by $\hbar \rightarrow 0$). This two kinds of statistics determine the two kinds of the commutation relations between the birth-annihilation operators (we must consider the commutation relations for the Bose statistics and the anticommutation relations for the Fermi statistics). The every physicist-theorist ought to know these relations ,therefore one can omit they here. But following is most important: the action of the birth and the annihilation operators onto the cyclical vector (in theoretical physics the cyclical vector is called *the vacuum vector*) .As it is usual in physics, let us denote the vacuum vector as $|0\rangle$ (*the 'bra' vector*). Then the action of the annihilation and the birth operators on $|0\rangle$ is expressed as

$$a(\Phi)|0\rangle = 0, a^*(\Phi)|0\rangle = \Phi \quad (0.1)$$

(5)

for the birth operators $a^*(\Phi)$ and the annihilation operators $a(\Phi)$.

The relation (2) and formula (7.99) in [3] show us that one can interpret the vectors of the Hilbert-Fock space as a *tensor products*: there are the *tensor functions of such number of momentum ,how much the particles we consider in our model*. But one can decompose the every tensor onto *the symmetrical and the antisymmetrical parts*. And what is more:as we know, the statistics determine the form of this tensor [4]: some symmetrical wave function for Bose statistics and some antisymmetrical wave function for Fermi statistics (it is in fact [8] the consequence of flag structure and spinor algebra,based on this structure).

If we denote the one-particle Hilbert space as a Λ_1 , then one can determine by the natural way the multiparticle space as $\Lambda_1^{\otimes n}$, - *the n^{th} (anti)symmetrical tensor power of the Hilbert space Λ_1* , which we will denote as $\Lambda_1^{\vee n}$ for Bose statistics and $\Lambda_1^{\wedge n}$ for Fermi statistics. As a particular case, by $n=0$ we have the scalar field with the usual scalar product $\Phi_0 \Psi_0$

The vectors of the states with different numbers of particles form the mutually-orthogonal subspaces in the complete Hilbert space. Hence, *it should to introduce the direct sum of the n -particle subspaces*. The direct sum

$$\bigoplus_{n=0}^{\infty} \Lambda_1^{\otimes n} \quad (6)$$

is called *the tensor algebra over the Hilbert space Λ_1* . The vectors of this space are the arbitrary sequences $\{\Phi_n\}_{n=0}^{\infty}$ such that $\Phi_n \in \Lambda_1^{\otimes n}$ and

$$\|\Phi\|^2 < \infty \quad (7)$$

and the scalar product has the form

$$\langle \Phi, \Psi \rangle = \sum_{n=0}^{\infty} \langle \Phi_n, \Psi_n \rangle \quad (8)$$

It is obvious that one can identify $\Lambda_1^{\otimes n}$ with the n-particle space in above direct sum. If $k \neq n$ then $\Phi_k = 0$. In particular, the one-dimensional 0-particle space $\Lambda_1^{\otimes 0}$ is called *the vacuum space*. It is 'pulled' onto the vector $\Phi_0 \equiv |0\rangle$ with its components

$$(\Phi_0)_0 = 1; (\Phi_0)_n = 1 \text{ by } n \neq 0 \quad (9)$$

which is called *the vacuum vector*. We can consider Φ_n as a projection of the sequence Φ on $\Lambda_1^{\otimes n}$. The vector Φ is called *the finite vector*, if it has *the finite number of its projections Φ_n different from zero*. It is obvious that the finite vectors form the linear manifold which is *dense* in the direct sum (6).

The direct sum (6) is divided with the natural way on the direct sum of its Bose and Fermi components, which we will denote as $F_V(\Lambda_1)$ and $F_\Lambda(\Lambda_1)$ correspondingly. These spaces are called *the Fock spaces of bosons and fermions* correspondingly.

The fundamental quantum-mechanical characteristics of the particle are *its mass and its spin*. We shall consider the one-particle space Λ_1 as a 'supplied' with these characteristics, and shall denote it as $\Lambda^{(m,s)}$. This space is transformed by the unitary representation of Poincare group (latter acts as an *automorphism* on $\Lambda^{(m,s)}$).

One can, of course, consider the spaces $\Lambda_1^{\vee n}$ and $\Lambda_1^{\wedge n}$ as *eigen-subspaces of the (Hermit) operator of the numbers of particles N*, which is determined on the (finite) vectors by formula

$$(N\Phi)_n = n\Phi_n \quad (10)$$

The construction of Fock space $F_V(\Lambda_1)$ or $F_\Lambda(\Lambda_1)$ by given space $\Lambda^{(m,s)}$ is called the second quantization

The birth and the annihilation operators allow the interpretation in the terms of the number of particles. Let us determine the symmetrical tensor product of vectors Φ, Ψ of space $F_V(\Lambda_1)$ which is *associative, distributive and commutative*. And analogous we determine the antisymmetrical tensor product of vectors Φ, Ψ of space $F_\Lambda(\Lambda_1)$ which is *distributive and associative only*.

$$\Phi \vee \Psi = \text{Sym } \Phi \otimes \Psi$$

$$\Phi \wedge \Psi = \text{Antisym } \Phi \otimes \Psi \quad (11)$$

Let us fix some vector Φ in formula (11) and let us consider then the maps

$$\Psi \rightarrow \sqrt{N}\Phi \vee \Psi \quad (12a)$$

or

$$\Psi \rightarrow \sqrt{N}\Phi \wedge \Psi \quad (12b)$$

The both maps are some linear operators on the finite vectors Φ of Fock (boson, fermion) spaces. The basic feature of these maps is that *the n-particle vector turns into the (n+1)-particle vector*, therefore it is called *the birth operator of the particle with wave function Φ* and denoted $a^*(\Phi)$. So, in the boson case the birth operator is determined with formula

$$a^*(\Phi) = \sqrt{N}\Phi \vee \Psi \quad (13a)$$

and analogous in the fermion case as

$$a^*(\Phi) = \sqrt{N}\Phi \wedge \Psi \quad (13b)$$

Then we can determine the annihilation operator as a operator conjugated to $a^*(\Phi)$. *This operator turns the n-particle vector in the (n-1)particle vector.*

Such is the briefly sketch of the second quantization theory. It is important that GNS construction of every Hilbert space $\Lambda^{(m,s)}$ generates the general structure of the Fock space.

Thus there exist the two interdependent approaches to the quantization problem : *the first quantization*, which is connected with replacement of Poisson brackets between canonical co-ordinates and canonical momentas onto the commutation relations with Planck consant \hbar . The second approach, *the second quantization*, is connected, as we just saw, with the structure of Fock- Hilbert space, with the two statistics and GNS construction.

The aim of this article is to show a some original variant of quantization: the quantization of the *particle velocity*. In fact, we will obtain *discrete Lorentz groupe*. This is one of examples of *quantum groups*, interest to which is very large now.

Mathematician V.G.Drinfel'd from Kharkow, apparently, is a pioneer in this sphere [9]. One can also recommend among the interesting works of this direction the work [10] of G.W.Delius, [11] of P. Schupp, [12] of M. Vybornov and many other on this theme.

Author wants to dedicate this work to respectful memory of his first teacher in theoretical physics- doctor V.M. Pyg, which deceased sudden in 1998 y. He was scientist ,well known with his works in the sphere of functional integration, conformnal gravitation, fibre bundles both in Soviet Union and far from its frontiers.

The invaluable help to author by the investigations, which preceded this work, rendered the employee of Kharkow low temperature Institute- doctor G.N. Geistrin: the well known Soviet and Ukrainian mathematician. He found the time for me in the difficult conditions of the 'postsoviet' period in the Ukraine; I am very grateful him for this.

II. The equation of movement for the scalar particle.

Let us [13] consider the charged scalar spinless particle in the electromagnetic field: the one of most simple theories. We set here $\hbar=c=1$. Then the movement equation for this particle is Klein-Gordon-Fock equation

$$[(\frac{\partial}{\partial x_m} - ieA_m)^2 - m^2]\Psi = 0 \quad (14)$$

Let us choose the 4-potential A_m as a

$$A_x = A_y = 0; A_z = -\frac{Ect}{2}; A_0 = -\frac{Ez}{2} \quad (15)$$

Such choice of potential sets the *constant electrical field along axis z*. We can, without loss of generality, work in the two dimensions (z,t). The only Lorentz invariant value:

$$u = z^2 - c^2t^2 \quad (16)$$

exists in such space.

Then we can find wave function $\Psi(z, t)$ in the form

$$\Psi(z, t) = G(v)F(u) \quad (17a)$$

where

$$v = z + ct \quad (17b)$$

This decomposition is quite correct. (The explicit writing down of values c and \hbar helps to see the

right dimensions of the physical values. We will later on, on in the case of need, to write down these values explicit). In particular, the dimension of the operator, which acts onto the wave function in (16) is the '*square of momentum*'.

Equation (16) allows the *standart Fourier method* of solution, i.e. the division of variables u and v .

It is easy to see that we have for $G(v)$ the very simple equation

$$\frac{v\dot{G}}{G} = \lambda \quad (18)$$

where λ is the *division parameter* and ' $\dot{}$ ' signifies the derivative by v . The solution of equation (18) is

$$G = \text{const } v^\lambda \quad (19)$$

And now let us solve the equation by u , which has the form

$$uF'' + (1 + \lambda)F' + \left(\frac{g^2}{16(\hbar c)^2} - \frac{k^2}{4} + \frac{ig\lambda}{4\hbar c}\right)F = 0 \quad (20)$$

where

$$k = \frac{mc}{\hbar} \quad (20a)$$

is *Compton wave vector* and

$$\mathbf{g} = e\mathbf{E} \quad (20b)$$

is the force acted onto the particle in the electrical field.

One can reduce this equation by means of some transformations to the *canonical form* of the *degenerate hypergeometrical equation* [14]

$$\theta Z''(\theta) + [(1 + \lambda) - \theta]Z'(\theta) - \left(-\frac{A}{\gamma}\right)Z(\theta) = 0 \quad (21)$$

where

$$A = (1 + \lambda)(\pm i\sqrt{C}) + b_2$$

$$b_2 = \frac{\lambda ig}{4\hbar c} - \frac{k^2}{4}; \quad \gamma = \mp 2i\sqrt{C} \quad (21a)$$

$$C = \frac{g^2}{16(\hbar c)^2} \quad (21b)$$

The connection of solution Z with solution F is following :

$$F = \exp(HU)Z \quad (22)$$

where

$$H = \pm i\sqrt{C} = \pm \frac{ig}{4\hbar c} \quad (22a)$$

There exist two linear-independent solutions of degenerate hypergeometrical equation (21). The first from them and most important is so-called *Whittaker- Pochhammer series* :

$$Z_1 = R\left(-\frac{A}{\gamma}; 1 + \lambda; \theta\right) = 1 + \frac{\theta}{1!} \frac{-\left(\frac{A}{\gamma}\right)}{1 + \lambda} + \frac{\theta^2}{2!} \frac{\left(-\frac{A}{\gamma}\right)\left(-\frac{A}{\gamma} + 1\right)}{(1 + \lambda)(2 + \lambda)} + \dots$$

$$+ \frac{\theta^n}{n!} \frac{\left(-\frac{A}{\gamma}\right)\left(-\frac{A}{\gamma} + 1\right)\left(-\frac{A}{\gamma} + 2\right)\dots\left(-\frac{A}{\gamma} + n - 1\right)}{(1 + \lambda)(2 + \lambda)\dots(n + \lambda)} \quad (23)$$

The second solution has the form

$$Z_2 = \theta^{1-(1-\lambda)} R\left(-\frac{A}{\gamma} - (1 + \lambda) + 1; 2 - (1 + \lambda); \theta\right) = \theta^{-\lambda} R\left(-\frac{A}{\gamma} - \lambda; 1 - \lambda; \theta\right) \quad (23a)$$

Let us consider solution (23). The basic demand to the wave function is its finiteness. That should to satisfy this demand we ought to *cut off* series (23). It is possible if *to set successive equal to zero* $-\frac{A}{\gamma}; -\frac{A}{\gamma} + 1\dots$. Thus we obtain the numerate system of the wave functions : it some discrete spectrum. This procedure of cutting off of series (23) is equivalent to *quantization of field* Ψ .

Let us consider firstly equation

$$\frac{\Delta}{\gamma} = 0 \quad (24)$$

Remembering designations (21) we obtain the equation for λ (λ is the division parameter).

$$-\frac{1+\lambda}{2} = \left(\frac{k^2}{4} - \frac{\lambda ig}{4\hbar c}\right) \frac{1}{\mp 2i\sqrt{C}} \quad (25)$$

We have the two cases; in the first case (25) has solution

$$\lambda = -\frac{1}{2} - \frac{ip}{2} \quad (26)$$

where

$$p = \frac{m^2 c^3}{g\hbar} \quad (26a)$$

is the dimensionless parameter.

We have in the second case the equation of type

$$\lambda 0 = t$$

i.e. $\lambda \rightarrow \infty$. This equation has no physical sense.

Now we obtain (with account of formulas (22), (26a), (19)) *namely the equation, which was proposed in work [12]:*

$$\Psi(z, t) = \exp(i \frac{z^2 - c^2 t^2}{\hbar c} \frac{q}{4}) (z + ct)^{-\frac{ip}{2} - \frac{1}{2}} \quad (27)$$

But the situation is more complicated and interesting than it see the authors of work [13]. If we equate $-\frac{\Delta}{\gamma} + 1 = 0$ etc. then we obtain the analogous spectrum ($\lambda_0, \lambda_1, \lambda_2 \dots$) and the corresponding wave functions. Thus solution (27) is some *zero approach only*.

Which a physical sense one can see in above equations? Variable u (look (16)) has the sense of the *(two)interval* and the expansion by eigenfunctions Ψ_n means the expansion by the *eigenfunctions of square of momentum* (the Fourier image of u is square of two-momentum).

The electrical field \mathbf{E} as we know from quantum electrodynamics [6] is the *source of the (virtual) electron-positron couples, or the other kinds of couples*. If n is the number of such couples then *is easy to compare our spectrum to such set of natural numbers*.

This allows the simple interpretation in terms of Feynman graphs. The physical mass of scalar field Ψ is indeed the sum of *its bare mass* and the *radiative correction* to the mass, which are provoked *with the interaction between the scalar and the other particles*. The interaction with the electron-positron couples is one of such interactions. This is the interaction of *electromagnetic type* because field Ψ has the electrical charge. One can construct the

effective lines, the vertex operators etc. for such theory. I want to state more in detail the position of the authors of work [13] on function (27) and its connection with the birth-annihilation processes in Appendix . This position is very interesting and important.

But we ought to consider the one consequence of solution (27) direct in this section. I quote here Yu.P. Stepanovsky and A.S. Bakaj- the authors of work [13]. The question is the *asymptotical behaviour* of function $\Psi(z, t)$ in (27).

The solution (27) describes the two kinds of particles: with the negative and with the positive charges. Really, potential A_0 in (15) becomes infinite and positive by $z \rightarrow -\infty$ and $t \rightarrow \infty$. Hence the probability for the positive charged particle to be in the points $z \rightarrow -\infty$ is very small. The other situation is at $z \rightarrow -t$ i.e. in the *ultra-relativistic region* near from light cone. Function

$\Psi(z, t)$ has its maximum in this region. Let us prove this.

Firstly, solution (19) has some constant. Let us choose as a such constant $\frac{mc}{\hbar}$ (with the dimension: 'length' $^{-1}$). Then we can rewrite the second factor in (27) as a

$$[(z + ct)\frac{mc}{\hbar}]^{-\frac{1}{2} - \frac{ip}{2}} = e^{(-\frac{1}{2} - \frac{ip}{2})\ln[(z+ct)\frac{mc}{\hbar}]} \quad (28)$$

(thus the index by the exponent is really dimensionless)

At $z \rightarrow -ct$ $\ln(z + ct) \rightarrow -\infty$. Therefore the probability for the particle to be in the light cone region increases as a *logarithm*. And the particle can to be even in the region $z \rightarrow -\infty$. Issuing from the potential distribution (15), we draw a conclusion that *solution (27), especially in the region $z \rightarrow -ct$ describes the particles of both charge signs.*

Takes place also the more interesting effect (although the author of this article don't knows for the present, how to interpret this phenomenon)

As it is well known, one can represent the real logarithm :

$$\ln(x) = \ln(x + i0y)$$

as a

$$\ln(x) + 2k\pi i$$

where $k \in \mathbf{Z}$.

Thus the *infinite degeneracy* arises by the real factors of $\exp(kp\pi)$ type in representation

$$e^{(-\frac{ip}{2} - \frac{1}{2})\ln[\frac{mc}{\hbar}(z+ct)]} = e^{(-\frac{ip}{2} - \frac{1}{2}) \operatorname{rmln}[\frac{mc}{\hbar}(z+ct)] + 2k\pi i} \quad (29)$$

Thus *our wave functions acquire in fact the constant extra factors of $\exp(kp\pi)$ type* and we have thus the *extra spectrum* of unknown for the time being nature which is *characterized with the set of quantum numbers k*

III. The Lorentz transformations.

Let us consider again solution (27) and let us subject the co-ordinates z and t to usual Lorentz transformation

$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad t' = \frac{t - \frac{vz}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (30)$$

where v is the velocity of the particle. Then

$$z' + ct' = (z + ct)\sqrt{\frac{c-v}{c+v}} \quad (31)$$

and wave function (27) with account of factor $\frac{mc}{\hbar}$ and with account of Lorentz -invariant vector \mathbf{E} (as it is well known from the theory of electromagnetic field [2], *if the direction of \mathbf{E} coincides with axis z which is subjected to the Lorentz transformation* then this is correctly) has form

$$\begin{aligned} \tilde{\Psi}(z', t') &= e^{\frac{-i(z'^2 - c^2 t'^2)g}{4\hbar c}} e^{(-\frac{ip}{2} - \frac{1}{2})\ln[\frac{mc}{\hbar}\sqrt{\frac{c-v}{c+v}}]} = \\ &= e^{\frac{-i(z'^2 - c^2 t'^2)g}{4\hbar c}} e^{-\frac{1}{2}\ln[\frac{mc}{\hbar}\sqrt{\frac{c-v}{c+v}}]} e^{-\frac{ip}{2}\ln[\frac{mc}{\hbar}\sqrt{\frac{c-v}{c+v}}]} \quad (32) \end{aligned}$$

Since $e^{-\frac{ip}{2} \operatorname{rmln}[\frac{mc}{\hbar}\sqrt{\frac{c-v}{c+v}}(z+ct)]}$ and $e^{-\frac{1}{2}\ln[\frac{mc}{\hbar}\sqrt{\frac{c-v}{c+v}}(z+ct)]}$ are the terms of solution (27) that the Lorentz transformation leads to the multiplication of solution (27) onto factor $\sqrt[4]{\frac{c-v}{c+v}}$ i.e. *onto the c -number* and hence to the correction of second exponent.

Let us impose the additional demand that *second exponent remains invariable by Lorentz trans-*

formation (30). This leads to the following equation

$$-\frac{p}{2} \ln \sqrt{\frac{c-v}{c+v}} = 2\pi k; k \in (0, \mathbf{Z}^+) \quad (33)$$

k is the nonnegative integer number since *the velocity is positive* and p (look (26a)) depends on the sign at $|g|$ which is '-' in case of negative charge e .

Equation (33) has the following solution

$$V_k = \frac{c[1 - e^{-\frac{8\pi k}{p}}]}{1 + e^{-\frac{8\pi k}{p}}} \quad (34)$$

Thus *the velocities form same discrete spectrum* and we now approach object of our work.

Exists the discrete Lorentz group which leads to the conformal transformations of the electrical charged field's wave function under which therefore the Lagrangian of the theory is invariant.

In fact we obtain *the quantum Lorentz group*. Why draw we such conclusion? Firstly, we have the numerical set of the permissible velocities. Secondly, the transformation (32) sets *the numerical co-ordinate representation of this Lorentz group*. We can, of course, interpret these sets of velocities and wave functions *as a sets of the Lorentz group eigenvalues and the eigenfunctions correspondingly*. Thus the quantization occurs *in fact without of Planck constant \hbar* (compare with article [10]).

Which are the basic features has this discrete Lorentz group (the fact that it is really a group our reader can prove easy without our assistance)?

So, it is *the discrete subgroup of general Lorentz group*, which is provoked with the special form of solution (17) of equation (14) with the definite choice of 4-potential (15). This is the *Abelian* group, because we have here the rotations in the one plane. As we just this saw, this is [3] *the discrete subgroup of eigen Lorentz group L_+^\uparrow* which is the *exactly* subgroup of general Lorentz group. The Lorentz boosts form the subgroup of L_+^\uparrow . Our discrete subgroup acts, on the other hand, as a *Weyl scale reparametrisation* multiplying all wave functions on the constant factor. Thus *our discrete Lorentz group, which acts as a Weil scale reparametrisations group is the subgroup of the boosts group*. We can interpret our above demand to the complex exponential factor in (32) to be the Lorentz invariant as a *gauge fixation which generate our discrete Lorentz group*. Such are the theoretical- group aspects of the obtained discrete Lorentz group.

Let us now consider the immovable frame. This frame allows us to observe the movement of the particle with the discrete set of velocities, i.e. we have *the set of eigenvalues of the velocity operator*. Our task now to construct such operator. Formula (34) *is the good prompt in the solution of this problem*.

Let us consider the following function :

$$f(x) = c \frac{1 - e^{-x}}{1 + e^{-x}} \quad (35)$$

and let us consider as a argument x *some operator \hat{v}* (thus we have also *the function of operator $f(\hat{v})$*) such that

$$\frac{8\pi}{p} \hat{v} = \frac{8\pi}{p} (c^2 t^2 \frac{\partial^2}{\partial z^2} - \frac{z^2}{c^2 t^2}) \quad (36)$$

It is obvious that \hat{v} is dimensionless and that, because of (34), its eigenvalues are the numbers $k \in [0, \mathbf{Z}^+]$. Thus we can write down

$$\frac{8\pi}{p}(c^2t^2\frac{\partial^2}{\partial^2z} - \frac{z^2}{c^2t^2})\tilde{\Psi} = \frac{8\pi}{p}k\tilde{\Psi}$$

or

$$(c^2t^2\frac{\partial^2}{\partial^2z} - \frac{z^2}{c^2t^2})\tilde{\Psi} = k\tilde{\Psi} \quad (37)$$

for some eigen-function $\tilde{\Psi}$ of the discrete Lorentz group representation. One can check that at $k=1$ $\tilde{\Psi} = \exp(\frac{-z^2}{2c^2t^2})$ is *first eigen-function of operator \hat{v}* .

Then we find the general solution of equation (37) in form

$$\tilde{\Psi} = e^{-\frac{\xi}{2}}\Theta(z), \quad (38)$$

where

$$\xi = \frac{z}{ct} \quad (38a)$$

It is easy to see that we so obtain, at the minimum formally, *the equation of harmonic oscillator*

$$(\frac{\partial^2}{\partial^2\xi} - \xi^2)\tilde{\Psi} = k\tilde{\Psi} \quad (39)$$

It is the equation of *Schrodinger type* and the method of solution of such equations is good described L.D. Landau and E.M. Lifshitz in their monograph [4]. We shall to follow farther this monograph in the statement of our method for equations (39).

This equation, in terms of ξ , Θ has form

$$\Theta''(\xi) - 2\xi\Theta'(\xi) + (k-1)\Theta(\xi) = 0 \quad (40)$$

This is *Hermit equation* at $k-1=2n$. But it is easy to show that *we obtain Hermit polynoms also at $k-1=n$ on the whole set \mathbf{N} of natural numbers*

Really, let us denote $k-1=n$; $2\xi = x$ or $\xi = \frac{x}{2}$. Then we obtain equation

$$\Theta''(\frac{x}{2}) - \frac{x}{2}\Theta'(\frac{x}{2}) + n\Theta(\frac{x}{2}) = 0 \quad (41)$$

This is [15] also the permissible form of Hermit equation.

Thus we can express the solution of equation (41) as a *family of Hermit polynoms H_n* with account of their normalisation

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) dx = 2^n n! \sqrt{\pi} \quad (42)$$

and the demand of orthonormality for wave functions $\tilde{\Psi}$. As a result we obtain

$$\tilde{\Psi}_n = \frac{1}{\sqrt{2^n n!}} \pi^{-\frac{1}{4}} \frac{1}{\sqrt{ct}} e^{\frac{-z^2}{2c^2t^2}} H_n(\frac{z}{ct}) \quad (43)$$

Thus we obtained the *spectrum of operator \hat{v}* with its values on set $k \in [0, \mathbf{Z}^+]$. But our interest is the *set of permissible velocities* in form (34), i.e. the *set of eigenvalues for the function of operator \hat{v}* :

$$F(\hat{v}) = \frac{c[1 - e^{-\frac{8\pi\hat{v}}{p}}]}{1 + e^{-\frac{8\pi\hat{v}}{p}}} \quad (44)$$

It is obviously that *this function is measurable simultaneously with numbers k i.e. the set of functions $\tilde{\Psi}$ is the set of eigenfunctions of the velocity operator.*

Take place the following equation which is correct by dimension reasons (we have the *differential operator of the dimension 'velocity' in left side of this equation*).

$$-c^3t^2\frac{\partial^2\tilde{\Psi}}{\partial^2z} = \frac{c[1 - e^{-\frac{8\pi\hat{v}}{p}}]}{1 + e^{-\frac{8\pi\hat{v}}{p}}} \tilde{\Psi} \quad (45)$$

The discrete nature of our Lorentz group prompts us also the following step of our investigations.

We decompose wave function $\tilde{\Psi}$ in *infinite series*

$$\tilde{\Psi} = \sum_i c_i \tilde{\Psi}_i \quad (46)$$

This is in fact the usual decomposition for the discrete spectrum in quantum mechanics (look

for example [4])). Thus we obtain again the *infinite-dimensional Hilbert space* .By analogy with *three-dimensional Decart space* we can say that $\tilde{\Psi}_i$ are the *orts* of the infinite-dimensional Hilbert space and $c_i\tilde{\Psi}_i$ are the corresponding *projections* of wave function $\tilde{\Psi}$ onto these 'directions' (again by analogy with usual analytical geometry). This is, in general, the appropriate geometrical interpretation for the discrete spectrum in quantum mechanics.

It is useful ,for better understanding of the matter, to remember briefly the features of projectors (look, for example monograph [16]).

So, let Hilbert space H disintegrates on direct sum

$$H = G \oplus F \quad (47)$$

of two Hilbert subspaces G and F .This means that these spaces are mutually-orthogonal .Therefore one can represent *every vector* $h \in H$ *as a unique sum*

$$h = g + f \quad (47a)$$

where $g \in G; f \in F$. These vectors g and f are called the *projections of vector* h *on subspaces* G *and* F *correspondingly*. And some operator which does it is called *the projector*. Let us denote this operator as P (or P_G , P_F when the question is the projection on the definite subspace).

Thus

$$g = P_G h \quad (48)$$

Operator P is *the linear operator* since Hilbert space is linear. This is also the *limited* operator with the *unit* norm. Really,

$$||h||^2 = ||g||^2 + ||f||^2$$

i.e.

$$||g|| \leq ||h|| \quad (49)$$

or

$$||P|| \leq 1 .$$

But if $h \in G$ then $g=h$, therefore we have the *equality* in (49). On the other hand, it is well known from the theory of operators [17] that *for every limited operator* P *which acts from the one normalised space into the other such space*

$$||P|| = \sup_{||x|| \leq 1} ||Px|| \quad (50)$$

for some vector $x \in H$. Since we can reach the equality in (49) then, because of above estimate for $||Px||$, operator P has really the unit norm.

The vector $g=Ph$ belongs to space G by every $h \in H$ therefore $Pg = g$, i.e. $P^2H = Ph$. Thus $P^2 = P$ (51)

Let us consider now some two arbitrary vectors $h_1, h_2 \in H$ and let

$$h_1 = g_1 + f_1; h_2 = g_2 + f_2$$

Then

$$(g_1, h_2) = (g_1, g_2) = (h_1, g_2)$$

i.e.

$$(Ph_1, h_2) = (h_1, Ph_2)$$

for every $h_1, h_2 \in H$. But this means that *operator* P *is the Hermitian operator*

$$P^* = P \quad (52)$$

And what is more : *this operator is positive*

$$(Ph, h) \geq 0 \quad (53)$$

Really

$$(Ph, h) = (P^2h, h) = (Ph, P * h) = (Ph, Ph) \geq 0$$

We can, of course, generalize this theory on the Hilbert space which is the direct sum of the arbitrary number of its subspaces (as we this for example saw in the case of the multi-particle Hilbert space). It is obvious also that

$$\sum_i P_i = 1 \quad (54)$$

for given direct sum of subspaces in Hilbert space H. The following feature of projectors is also obvious .

If two arbitrary subspaces G_1 and G_2 of Hilbert space H are mutually- orthogonal then

$$P_{G_1} P_{G_2} = 0 \quad (55)$$

If H is ortho-normalised Hilbert space then it is broken up on the *infinite direct sum of its vectors* which corresponds to *the infinite sum of one-dimensional projectors with above features*.

And let us now consider, how above theory of projectors affects on the *theory of self-conjugate operators*. As we this known (for example from the course of quantum mechanics) such operator *has the real spectrum*. Our operator of velocity is one of such operators.

Let now [3] consider *Hilbert space $\mathcal{H}^2(X; \mu)$ of the complex measurable functions with the integrable square and with measure μ on space X.*

The every (real or complex) measurable function sets ,obviously, it the operator of multiplication on α on space X. Let latter is determined on set D_α . Then we understand *isomorphism V between Hilbert spaces H and $\mathcal{H}^2(X; \mu)$ with feature $A = V^{-1}\alpha V$ as a realisation of (self-conjugate) operator A in Hilbert space H*. We also mean that $D_\alpha = VD_A$ for the set of definition D_A of operator A.

The spectral theorem affirms that every self-conjugate or unitary operator A in Hilbert space H can be realised with the operator of multiplication in appropriate Hilbert space $\mathcal{H}^2(X; \mu)$.

We can impart the alternative form to above theorem in terms of considered us theory of projectors.

The feature (54) of projectors is called *the decomposition of unit* in the theory of projectors .

If operator A is realised as an operator of multiplication on the *real function* in $\mathcal{H}^2(X; \mu)$ then one can introduce *some family of the multiplication operators e_λ in $\mathcal{H}^2(X; \mu)$ of the functions of real parameter λ which determined with equalities*

$$e_\lambda(x) = 1 \text{ by } \alpha(x) < \lambda$$

and

$$e_\lambda(x) = 0 \text{ by } \alpha(x) \geq \lambda \quad (56)$$

It is obviously that e_λ are the *projectors* in $\mathcal{H}^2(X; \mu)$; therefore formula

$$E_\lambda = V_{-1} e_\lambda V \quad (57)$$

determines *the family of (orthogonal) projectors in H which depends also on λ .*

It is easy to certain in truth of following features of projectors:

$$a. \quad E_\lambda E_\mu = E_\lambda \text{ by } \lambda \leq \mu \quad (58a)$$

We obtain formula (55) as its particular case for the *orthogonal projectors*.

$$b. \lim_{\lambda \rightarrow -\infty} E_\lambda \Phi = 0, \lim_{\lambda \rightarrow \infty} E_\lambda \Phi = \Phi, \lim_{\lambda \rightarrow \mu-0} E_\lambda \Phi = E_\mu \Phi \quad (58b)$$

for every $\Phi \in H, \mu \in \mathbf{R}$.

c. The following integral representation takes place (as a *Stieltjes integral*)

$$A\Phi = \int_{-\infty}^{\infty} \lambda dE_\lambda \Phi \quad (58c)$$

(We want to remind here our reader that the *Stieltjes-(Riemann) integral of the function f(x) with the integrable function g(x) on the (limited) interval[a,b] is ,by definition,*

$$\int_a^b f(x)dg(x) = \lim_{\max(x_i - x_{i-1}) \rightarrow 0} \sum_{i=1}^m f(\xi_i)[g(x_i) - g(x_{i-1})]$$

where

$$a = x_0 < x_1 < x_2 < \dots < x_m = b \quad (a)$$

and $x_{i-1} \leq \xi_i \leq x_i$ If $g(x)$ is *some function of limited variation,i.e. if exists the such positive number M that for division (a) of interval [a,b] the following equation is true :*

$$\sum_{i=1}^n |g(x_i) - g(x_{i-1})| < M$$

and $f(x)$ *is the such continuous function on [a,b] that above limit exists.*One can easy generalise the definition for interval [a,b] on the case *of the infinite interval*, as one makes that in the usual integral calculus.

This theory is quite suitable also for our case of the discrete (*numerical*) spectrum because of the limit relations in above definition.

The formula (58c) is called the *spectral decomposition of self-conjugate operator A*.

Issuing from above spectral theory we can now write down the following definition for the *function of self-conjugate operator A* [17]

The function $\phi(A)$ is the operator which is determined by formula

$$\phi(A)f = \int_{-\infty}^{\infty} \phi(\lambda)dE_\lambda f \quad (59)$$

for all vectors $f \in H$ for which formula (59) is true .

And now we can ,in conclusion, complete our consideration of the discrete Lorentz group theory. We can *compare real parameter λ to the discrete spectrum of velocities*. In fact ,with accounting of the limit nature of formula (58c) in sense of Stieltjes- Riemann integral,we have now the following situation.

We showed already that decomposition (46) gets us the representation of some vector $\tilde{\Psi}$ of (ortho-normalised) Hilbert space H by its co-ordinates $\tilde{\Psi}_i$. Then the substitution of function $\tilde{\Psi}$ in formula (59) and the comparison of parameter λ with the set of factors c_i in formula (46) *give us the correct theory for the differential operator of velocity (equation (45)) and for its wave function $\tilde{\Psi}$.*

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Appendix. The some features of quantization for the scalar field in presence of the homogeneous electric field.

We finish our article with the statement of doctors Y.P.Stepanovsky and A.S.Bakaj original point of view on the birth -annihilation processes in the scalar field electrodynamics. Their theory is connected closely with the theory considered in this article, and it is very didactic,-to expound it here .

So, our careful analysis of solution (27) by choice (15) of 4-potential A_μ showed that the points with co-ordinates $z \rightarrow -\infty$ are *inaccessible* at $t \rightarrow \infty$ for the particles with the positive charges. And, on the contrary, ultra-relativistic region $z \rightarrow -ct$ is the region *accessible for the particles of the both sign of charge* .

The value of vawe function $\Psi(z, t)$ at $\frac{m^2 c^3}{eE\hbar} \gg 1$ in above classical (on level of *classical electrodynamics*) inaccessible region for the positive charged particles is *much less than its value in the classical accessible points for the positive charged particles* $t \rightarrow -\infty, z \rightarrow \infty$

$$\frac{\Psi(-z, t)}{\Psi(z, -t)} = (-1)^{-i\frac{m^2 c^3}{eE\hbar}} = (e^{-i\pi})^{-i\frac{m^2 c^3}{2eE\hbar}} = e^{-\frac{\pi m^2 c^3}{2eE\hbar}} \quad (A1)$$

But latter value becomes *much more than unit* if we adopt $-1 = \exp(i\pi)$. And this is equal to the *charge conjugated theory*.

If we consider wave function (27) as a function of the complex variable $W = iet + z$ then one can pass from the negative to the positive z through point $W = 0$, - *the point of branching out of this variable*. And we obtain the *different* signs in formula $-1 = \exp(\pm i\pi)$ depend on the direction of the roundabout way of the branching out point.

Let us such modify function (27) :

$$\Psi(x, t) = e^{ipx - i\mathcal{E}t} e^{\pm i \frac{c[(p_z + \frac{eE}{2}t)^2 - (\mathcal{E} + \frac{eE}{2}z)^2]}{\hbar eE}} * \\ * [(p_z + \frac{eE}{2}t) \mp (\mathcal{E} + \frac{eE}{2}z)]^{\pm \frac{p_x^2 - p_y^2 + m^2 c^3}{2\hbar eE} - 1} \quad A(2)$$

where $p_x, p_y, p_z, \mathcal{E}$ are the values of projection of momentum and energy correspondingly; i.e. this function depends on the general 4-vector of momentum in which homogeneous electric field (15) is joined. Then formula (A1) turns into

$$\frac{\Psi(x, y, z, t)}{\Psi(x, y, -z, t)} = e^{-\pi \frac{p_x^2 - p_y^2 + m^2 c^3}{2\hbar eE}} \quad (A3)$$

We assume now that the electric field is 'engaged' at $t \rightarrow -\infty$ and 'switched off' at $t \rightarrow \infty$. This field will inevitably generate the *charge-conjugate* couples :the scalar bosons of both signs, the electron-positron, quark-antiquark and the other couples.

We consider now the birth with this field of the *scalare spinless couple of both charges*. And let us consider the such process when *the positive-frequency wave function* $\exp(i\mathbf{p}_k \mathbf{x} - i\mathcal{E}_k t)$ *turn into the superposition of positive and negative-frequency wave functions*

$$e^{i\mathbf{p}_k \mathbf{x} - i\mathcal{E}_k t} \rightarrow f e^{i\mathbf{p}_k \mathbf{x} - i\mathcal{E}'_k t} + g e^{i\mathbf{p}_k \mathbf{x} + i\mathcal{E}'_k t} \quad (\text{A4})$$

Let analogous

$$e^{i\mathbf{p}_k \mathbf{x} + i\mathcal{E}_k t} \rightarrow f^* e^{i\mathbf{p}_k \mathbf{x} + i\mathcal{E}'_k t} + g^* e^{i\mathbf{p}_k \mathbf{x} - i\mathcal{E}'_k t} \quad (\text{A5})$$

Then the general solution of equation (14) *in terms of the negative and positive charged particles operators* (in Fock space of second quantization) which has the standart form

$$\Psi(x, t) = \sum_k \frac{1}{\sqrt{2V\mathcal{E}_k}} (a_k e^{i\mathbf{p}_k \mathbf{x} - i\mathcal{E}'_k t} + b_k e^{i\mathbf{p}_k \mathbf{x} + i\mathcal{E}'_k t}) \quad (\text{A6})$$

turns into

$$\Psi'(x, t) = \sum_k \frac{1}{\sqrt{2V\mathcal{E}_k}} (f a_k + g^* b_k^+) e^{i\mathbf{p}_k \mathbf{x} - i\mathcal{E}'_k t} + (f^* b_k^+ + g a_k) e^{i\mathbf{p}_k \mathbf{x} + i\mathcal{E}'_k t} \quad (\text{A7})$$

where a_k is the annihilation operator for the particle with momentum \mathbf{p}_k and b_k^+ is the birth operator for the particle with momentum $-\mathbf{p}_k$. Therefore we can interpret the factors at the exponent in (A7) as a correspondingly *birth and annihilation operators in electric field* (15):

$$a'_k = (f a_k + g^* b_k^+) \sqrt{\frac{\mathcal{E}'_k}{\mathcal{E}_k}} \quad (\text{A8.a})$$

and

$$b'_k = (f^* b_k^+ + g a_k) \sqrt{\frac{\mathcal{E}'_k}{\mathcal{E}_k}} \quad (\text{A8.b})$$

It is obvious that the *vacuum average of the operators of number of particles* $:a_k'^+ + a'_k$ and $b_k'^+ b'_k$ *is different from zero*:

$$N_k = \langle 0 | a_k'^+ + a'_k | 0 \rangle = \langle 0 | b_k'^+ + b'_k | 0 \rangle = \frac{\mathcal{E}'_k}{\mathcal{E}_k} |g|^2 \quad (\text{A9})$$

where the definition of vacuum (5), its norm : $\langle 0 | 0 \rangle = 1$ and the commutation relations between the birth and annihilation operators are taken into account.

Formula (A9) shows us that *we should to solve classical equation (14) and to find then factorg from equation (A4) (this effect of the mixing up of the positive and the negative frequencies in formula (A8) is called Klein para-*

dox). One can show that this task in case of the homogeneous electric field 'engaged' at $t \rightarrow -\infty$ and 'switched of' at $t \rightarrow \infty$ comes to the *task about the oscillator with the variable frequency*.

Let us denote the homogeneous electric field dependence on time as $\mathbf{E}(t)$. This field is generated with the vector-potential $\mathbf{A}(t)$: $\mathbf{E}(t) = -\frac{\partial \mathbf{A}}{\partial t}$ (A10)

If we represent the solution of equation (14) in the form

$$\Psi(x, t) = \xi(t) e^{i\mathbf{P}\mathbf{x}} \quad (\text{A11})$$

then we obtain *the oscillator equation* relatively variable ξ :

$$\ddot{\xi} + \omega^2(t) \xi = 0 \quad (\text{A12})$$

where

$$\omega^2(t) = (\mathbf{P} - e\mathbf{A}(t))^2 + m^2 \quad (\text{A13})$$

Let us consider now the case when field $\mathbf{E}(t)$ is directed along axis Oz and has the following dependence on time :

$$E_z = \frac{E}{\text{ch}^2 \frac{2t}{T}} \quad (\text{A14})$$

The vector-potential

$$A_z = -\frac{ET}{2} \text{th} \frac{2t}{T} \quad (\text{A15})$$

The parameter T which we shall rush to infinity one can interpret as *an average time of the action of field E*, since

$$\int_{-\infty}^{-\infty} E_z(t)dt = A_z(-\infty) - A_z(\infty) = Et \quad (\text{A16})$$

Substituting potential (A15) in (A13) und supposing that $P_z = p_z$ and $P_y = p_y$ we obtain from equation (A12)

$$\ddot{\xi} + (Ath^2 \frac{2t}{T} + Bth \frac{2t}{T} + C)\xi = 0 \quad (\text{A17})$$

where

$$A = \frac{(eET)^2}{4}, B = eETP_z,$$

$$C = p_x^2 + p_y^2 + P_z^2 + m^2 \quad (\text{A17a})$$

This is formally equation of oscillator. Let us denote

$$\frac{2}{T} = \varepsilon \quad (\text{A18})$$

i.e. we can rewrite equation (A17) as

$$\ddot{\xi} + (Ath^2 \varepsilon t + Bth \varepsilon t + C)\xi = 0 \quad (\text{A17c})$$

(ε has the physical sense of the energy of oscillator).

It is turn out that one can transform the such equation of oscillator to hypergeometrical equation

$$[\eta(\eta-1)\frac{d^2}{d\eta^2} + ((\alpha+\beta+1)\eta-\gamma)\frac{d}{d\eta} + \alpha\beta]F = 0 \quad (\text{A19})$$

where

$$\eta = e^{2\varepsilon t} \quad (\text{A19a})$$

$$\alpha = \mu + \lambda + i \frac{\sqrt{A+B+C}}{2\varepsilon}$$

$$\mu = \pm i \frac{\sqrt{A-B+C}}{2\varepsilon},$$

$$\lambda = \frac{1}{2} \pm \frac{1}{2} \sqrt{-1 - \frac{4A}{\varepsilon^2}},$$

$$\beta = \mu + \lambda - i \frac{\sqrt{A+B+C}}{2\varepsilon},$$

$$\gamma = 1 + 2\mu \quad (\text{A19b})$$

(We made here the substitution

$$\xi(t(\eta)) = (-\eta)^\mu (1-\eta)^\lambda F(\alpha, \beta, \gamma, \eta))$$

One can represent solution F as a *hypergeometrical series*

$$F(\alpha, \beta, \gamma, \eta) = 1 + \frac{\alpha\beta}{1\cdot\gamma}\eta + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot\gamma(\gamma+1)}\eta^2 + \dots, \quad (\text{A20})$$

If α and β are different from 0,-1,-2,...(the serie is broken at these values) and $\gamma \neq 0, -1, -2, \dots$ then *series (A20) converges absolutely at all $|\eta| < 1$* . According to (A19a) $\eta \rightarrow 0$ at $t \rightarrow -\infty$ (we consider parameter ε as a positive parameter). Thus at $t \rightarrow -\infty$ the solution of equation (A17c) has the form

$$\xi(t(\eta)) = (-\eta)^\mu \quad (\text{A21})$$

(we utilize here the *majorant* estimate for the sum of the series).

We *postulate* now that constants A,B,C in equation (A17c) are *connected with the asymptotical values of the oscilator frequencies* : $\omega(\pm\infty) \equiv \omega_{\pm}$ and $\omega(0) \equiv \omega_0$:

$$A = \frac{\omega_+^2 + \omega_-^2}{2} - \omega_0^2,$$

$$B = \frac{\omega_+^2 - \omega_-^2}{2}, C = \omega_0^2 \quad (\text{A22})$$

Then we can rewrite formula (A21) (with account of denotations (A19)) as

$$\xi(t(\eta)) = (-\eta)^\mu = e^{\pm i\omega_- t} \quad (\text{A21a})$$

Let us ascertain how looks the same solution at $t \rightarrow \infty$ when $\eta \rightarrow \infty$. As we this known from the theory of hypergeometrical functions[14]

$$F(\alpha, \beta, \gamma, \eta) = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\eta^{-\alpha}F(\alpha, 1-\gamma+\alpha; 1-\beta+\alpha; \frac{1}{\eta}) + \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\eta^{-\beta}F(\beta, 1-\gamma+\beta; 1-\alpha+\beta; \frac{1}{\eta}) \quad (\text{A23})$$

According to (A23) at $t \rightarrow \infty$

$$\xi(t(\eta)) \approx \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\eta^{\mu+\lambda-\alpha} + \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\eta^{\mu+\lambda-\beta} \quad (\text{A24})$$

or, in terms of the asymptotical frequencies :

$$\xi(t) \approx \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}e^{-i\omega_+t} + \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}e^{i\omega_+t} \quad (\text{A25})$$

The following formula is quite correct

$$e^{-i\omega_-t} \rightarrow \sqrt{\frac{\omega_-}{\omega_+}}(fe^{-i\omega_+t} + ge^{-i\omega_+t}) \quad (\text{A26})$$

We can see, even 'with the naked eye', that this formula is *fit* for our model of the homogeneous electric field 'engaged' at $t \rightarrow -\infty$ and 'switched off' at $t \rightarrow \infty$. Let this change of the field is *very slow*. We [1] imply under the word 'slow' that

$$T_1 \frac{d\lambda}{dt} \ll \lambda \quad (\text{A27})$$

for the field parameter λ and the period T_1 . Such slow change of the field is called *the adiabatical change* of the field. The very important characteristics of the *adiabatical process* are the *values which are invariant* by such process. They are called *the adiabatical invariants*.

Let us calculate the such adiabatical invariants for theory (A25) and thus we shall obtain the correct model *for the slow changed homogeneous electric field*.

We can mark that the explicit expression for the factors f and g from formula (A26) we have from formula (A25).

One can represent the arbitrary (real) solution of oscillator equation at

$t \rightarrow -\infty$ as

$$\xi(t) = \sqrt{\frac{2}{\omega_-}} \text{Re}(a_- e^{-i\omega_-t}) \quad (\text{A28})$$

We can apply above strategy also to the solution at

$t \rightarrow \infty$ and write down

$$\begin{aligned} \sqrt{\frac{2}{\omega_-}} \text{Re}(a_- e^{-i\omega_-t}) &\rightarrow \sqrt{\frac{2}{\omega_+}} \text{Re}(a_- e^{-i\omega_+t}) = \\ &= \sqrt{\frac{2}{\omega_+}} \text{Re}(fa_- e^{-i\omega_+t} + ga_- e^{-i\omega_+t}) = \sqrt{\frac{2}{\omega_+}} \text{Re}[fa_- + g * a_-] e^{-i\omega_+t} \end{aligned} \quad (\text{A29})$$

As we known from the course of theoretical mechanics [1], the integral

$$I = \frac{1}{2\pi} \oint p dq \quad (\text{A30})$$

in the phase space is *the adiabatical invariant*, i.e.

$$\frac{dI}{dt} = 0 \quad (\text{A31})$$

One can prove that for the harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \quad (\text{A32})$$

where p - is the momentum of oscillator; m is its mass; ω is its frequency and q is its co-ordinate.

It turns out that

$$I = \frac{E}{\omega} \quad (\text{A33})$$

where E is the energy of the oscillator. It is easy to see that in our case

$$I = \frac{\varepsilon}{\omega} = |a|^2 \quad (\text{A34})$$

Really, since $|a|^2$ has the quantum-mechanical sense of *probability* for some quantum-mechanical value, then formula (A33) means *that this probability $|a|^2$ is in fact constant*. Also if one looks

onto expression (A29) that one can mark that average of $e^{-i\omega-t}$ is equal to 1. Therefore the addiabatical invariant of the oscillator should to coincide with $|a|^2$.

Then from (A34) and (A29) we obtain the formula for the change of the addiabatical invariant I $|a_-|^2 \rightarrow |fa_- + g * a_-|^2$ (A35)

Hence

$$\frac{\Delta I}{I} = \frac{I_+ - I_-}{I_-} = \frac{|fa_- + g * a_-|^2}{|a_-|^2} \quad (\text{A36})$$

since the Wronscian

$$\Psi^* \dot{\Psi} - \dot{\Psi}^* \Psi = \text{constant}$$

for the two complex-conjugate solutions of (A28),(A29) then

$$|f|^2 - |g|^2 = 1 \quad (\text{A36})$$

Introducing the denotation

$$\frac{|g|^2}{|f|^2} \equiv \rho \quad (\text{A37})$$

and utilising (A36) we obtain

$$f = e^{i\delta_1} \frac{1}{\sqrt{1-\rho}}, g = e^{i\delta_2} \sqrt{\frac{\rho}{1-\rho}} \quad (\text{A38})$$

where δ_1 and δ_2 are the phases of f and g correspondingly. Substituting (A38) into (A36) and representing a as $a = |a|e^{i\delta}$ we obtain

$$\frac{\Delta I}{I} = \frac{2 \cos \phi \sqrt{\rho} + 2\rho}{1-\rho} \quad (\text{A39})$$

where

$$\phi = 2\delta + \delta_1 + \delta_2 \quad (\text{A39a})$$

Thus *the exactness of the conservation of the addiabatical invariant I depends on the value of ρ and the phase ϕ of the harmonic oscillation which depends on the phase δ at $t \rightarrow -\infty$.*

Since the formal equation for the harmonic oscillator : $\ddot{x} + \omega^2 tx = 0$ coincides with *one-dimensional Schrodinger equation* then it is useful to interpret ρ as a *factor of the 'over-barrier' reflection* and to utilize in this case *the quasi-classical approach* for the calculation of ρ .

According to (A25)

$$\rho = \left| \frac{\Gamma(\beta)\Gamma(\gamma-\alpha)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} \right|^2 \quad (\text{A40})$$

Utilizing the features of Γ -functions and the fact that constants ω_+, ω_- and ω_0 are real, one can find that

$$\rho = \frac{\text{ch}[\pi(\omega_+ - \omega_-)/\varepsilon] + \cos(\pi\sqrt{1-4A/\varepsilon^2})}{\text{ch}[\pi(\omega_+ + \omega_-)/\varepsilon] + \cos(\pi\sqrt{1-4A/\varepsilon^2})} \quad (\text{A41})$$

where, in according with (A22)

$$\omega_{\pm} = \sqrt{A \pm B + C} \quad (\text{A41a})$$

The frequency $\omega(t)$ is not equal to zero if

$$4A < (\omega_+ + \omega_-)^2 \quad (\text{A42})$$

Let us now return to equation (A17). If $T \rightarrow \infty$ and $|P_z| < \frac{EeT}{2}$ then formula (A41a) yields

$$\hbar\omega_{\pm} \approx c \left[\frac{EeT}{2} \pm P_z + \frac{p_x^2 + p_y^2 + m^2}{eET} \right] \quad (\text{A43})$$

(we wrote out here the correct dimension' (*energy*')). Then (again in the system $\hbar = c = 1$) we have the inequality

$$(\omega_+ - \omega_-)^2 < 4A < (\omega_+ + \omega_-)^2 \quad (\text{A44})$$

Utilising (A9) and (A38) we obtain for the average of the operator of number of particles

$$N_k = \frac{\mathcal{E}'_k}{\mathcal{E}_k} |g|^2 \approx \frac{\rho}{1-\rho} \quad (\text{A45})$$

We ought now to estimate the value of ρ .

If the condition (A44) is carried out that at $\epsilon \rightarrow 0$ we have the following asymptptical estimates for ρ

$$\begin{aligned} \rho &= e^{-\frac{2\pi\omega_-}{\epsilon}} (\omega_- \leq \omega_+) , \\ \rho &= e^{-\frac{2\pi\omega_-}{\epsilon}} (\omega_+ \leq \omega) \quad (\text{A46}) \end{aligned}$$

if

$$4A < (\omega_+ - \omega_-)^2, \quad (\text{A46a})$$

and

$$\rho = e^{-\frac{\pi[\omega_+ + \omega_- - 2\sqrt{A}]}{\epsilon}} \quad (\text{A47})$$

if

$$4A \geq (\omega_+ - \omega_-)^2 \quad (\text{A47a})$$

Then inequality (A44) leads to estimate (A47a) for ρ and to formula (A47). The substitution of the values of ω_+ and ω_- in formula (A47) for ρ yields the result

$$\rho = e^{-\pi \frac{p_x^2 + p_y^2 + m^2}{eE}} \quad (\text{A48})$$

The values of ρ are exponential small at $m^2 \gg eE$, i.e. if field E is *adiabatical*. One can neglect the value of ρ in denominator of (A45) in this case and the value of ρ defines the average of the borned couples which coincides with the probability of the birth of the one couple because of above trifle of

ρ . One can quote here the general formula for the probability of the birth of the n couples (the above probability of the birth of the one couple, (A48), is the particular case of this general formula):

$$W_n = (1 - \rho)\rho^n \quad (\text{A49})$$

According to (A49) the exact sense of the value of ρ is determined on the following way: the value $W_0 = 1 - \rho$ is the probability of the following event:

'the no couple with the appropriate quantum numbers is borned in the vacuum'. But as it is well known from the course of quantum mechanics (look, for example, monograph [4], paragraph 41) *the quantum-mechanical state with the definite quantum numbers is (in fact) the invariant by the adiabatical evolution of the quantum system*. This statement, which was proved with the help of the mathematical apparatus of the *quantum perturbative theory* is called *P.Ehrenfest hypothesis*. Thus, in terms of P.Ehrenfest hypothesis, the value of ρ determines *non-fulfilment of P.Ehrenfest hypothesis*.

Let us find *the full number of couples which are borned with electric field E at $m^2 \gg eE$* . Let us multiply for this the value of ρ onto the element of the phase volume $V dp_x dp_y dP_z / (2\pi\hbar)^3$ and let us integrate over the all permissible values of momentum (ρ has the sense of the density of probability). It is necessary to integrate over dP_z in the limits from $-EeT/2$ to $EeT/2$. It is connected with the truth of formula (A48) for ρ at $|P_z| < \frac{eET}{2}$. The analysis of formula (A47) at $|P_z| > \frac{eET}{2}$ and $T \rightarrow \infty$ shows us that $\rho = 0$ in this case. After the integration (as a Gauss integral) we obtain

$$\begin{aligned}
N &= \frac{V}{(2\pi\hbar)^3} \int e^{-\pi \frac{p_x^2 + p_y^2 + m^2}{eE}} dp_x dp_y dP_z = \\
&= VT \frac{(eE)^2}{(2\pi\hbar)^3} e^{\frac{-\pi m^2}{eE}} \quad (\text{A50})
\end{aligned}$$

Thus the average number of the couples borned with the addiabatical changed homogeneous electric field E in the unit of the volume during the unit of the time is

$$n = \frac{(eE)^2}{(2\pi\hbar)^3} e^{\frac{-\pi m^2}{eE}} \quad (\text{A51})$$

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